JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 22, No. 6, November–December 1999

Multiple-Spacecraft Orbit Transfer Problem: Weak-Booster and Strong-Booster Cases

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Primer vector theory, in combination with a parameter optimization algorithm, is used to compute the optimal transfer of n spacecraft from an initial parking orbit to a final operational orbit with the added constraint that all spacecraft are injected from the parking orbit on one upper-stage booster and that all are required to be spaced along the final orbit according to some prescribed, but otherwise arbitrary, spacing constraint between individual spacecraft. Two particular cases of the problem, known as the weak- and the strong-booster cases, are examined. In the weak-booster case, the booster provides the departure maneuver from parking orbit, but because of a booster propellent constraint, its trajectory does not reach the final orbit, requiring that all spacecraft make at least two maneuvers to complete the transfer. For the strong-booster case, the booster departure maneuver is unconstrained in magnitude, implying that at least one of the spacecraft can complete the transfer with a single maneuver because the transfer trajectory can reach the final orbit. The solution is formulated for a general force field, and examples are given for a three-spacecraft constellation transfer in the restricted three-body problem force-field model.

Introduction

THE current study is a continuation of previous results[1] that deal with computing and optimizing transfer solutions for multiple spacecraft. The problem has been termed the single-booster multiple-spacecraft transfer problem and is subdivided into three distinct cases: no booster, weak booster, and strong booster.

Case I, no booster: This case considers the transfer of n spacecraft between two arbitrary orbits without the aid of an upper-stage booster. All of the spacecraft move from the initial orbit O^0 to the final orbit O^f, subject to the final spacing constraint, and the booster does nothing.

Case II, weak booster: In this case, all spacecraft are injected onto an intermediate transfer orbit O^m with the aid of an upper stage booster. After injection, all spacecraft perform the number of impulses required for moving from the transfer orbit to the final mission orbit. The primary assumption for this case is that the intermediate trajectory does not intersect the final orbit, and thus all spacecraft are required to perform at least two maneuvers to complete the transfer. This case results when a propellant constraint is imposed on the booster, thus bounding the maximum injection Δv capability for a given payload. As a consequence, the transfer orbit may not reach the final orbit and hence the designation as the weak-booster case. This case is used when the upper-stage booster parameters are fixed and it is required to determine the optimal conditions for the transfer of the n spacecraft by using their own dedicated propulsion systems to move from O^m to O^f.

Case III, strong booster: This case is similar to case II without the booster propellant constraint, therefore making it possible for the booster transfer orbit to intersect the final orbit at least once, so that at least one spacecraft can complete the transfer with a single impulse. If the injection orbit intersects the final orbit more than once, as is the case if it is periodic, for example, then it may even be possible for all spacecraft to perform a single impulse to complete the transfer. This case is used when it is desired to size the upper-stage booster for a given payload of n spacecraft. In this case, the booster and the first spacecraft move from O^0 to O^f by means of O^m and the remaining spacecraft move from O^m to O^f.

Cases II and III are treated in this paper. A cost is defined, and the differential of the cost is derived in the form of a cost gradient such that the coefficients of the independent variations are the elements of the gradient vector for an unconstrained optimization algorithm. Further examination of each coefficient provides information about the conditions that must be satisfied on a stationary solution to the transfer problem. The value and the sign of these coefficients also provide information on how a nominal solution can be improved at least to first order by a slight adjustment of the independent variables in the right direction.

The solution to this problem is needed for the optimization of future satellite constellation transfer problems. Although the examples assume a restricted three-body problem (RTBT) force-field model,[2] these can be extended and applied to any force field for which the solution to this problem is required. The classes of orbits used in the numerical examples are known as distant Earth orbits, which are ideal orbits for future astronomical satellites.[3] The application of primer vector theory to the development of the necessary conditions for optimal impulsive orbit transfers between RTBP trajectories and, in particular, distant Earth orbits has been examined in a recent study.[3] In general, the use of primer vector theory for impulsive and finite burn maneuvers has been examined and documented extensively in the literature,[4–8] and these results form the basis of the current study.

Case II: The Weak-Booster Case

The weak-booster case considers the injection of n spacecraft into an intermediate orbit that does not intersect the final target orbit. This is the case when the $\Delta v_{0\,\max}$ capability of the booster is bounded because of a propellant constraint and the resulting transfer orbit cannot reach the final orbit. The booster and the n spacecraft are initially on the orbit denoted by O^0. The intermediate transfer orbit is referred to as O^m and all n spacecraft then perform the required maneuvers to move from O^m to O^f. The initial payload (the n spacecraft) on O^0 is specified, as are the characteristics of the booster stage.

For a given booster and payload mass, the maximum Δv capability available from the booster injection maneuver is given by

$$\Delta v_{0\,\max} = c_B \ln\left(\frac{m_{\text{payload}} + m_{\text{stage}} + m_{\text{fuel}}}{m_{\text{payload}} + m_{\text{stage}}}\right)$$

where c_B is the booster engine exhaust velocity, m_{stage} is the booster stage mass, m_{fuel} is the booster propellant mass, and m_{payload} is the payload mass given by the n spacecraft:

$$m_{\text{payload}} = n m_0^i$$

Received 18 December 1997; revision received 23 December 1998; accepted for publication 26 February 1999.

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where m_0^i refer to the initial individual spacecraft masses including their own dedicated fuel and propulsion system masses. This is equivalent to the spacecraft's separated mass. The booster injection maneuver, described by the departure location from O^0 and the three components of $\Delta \boldsymbol{v}_0$, is included in the overall optimization process. The cost of the boost maneuver in terms of propellant consumed does not form part of the cost function. However, the cost in terms of the total Δv of the n spacecraft will depend on the parameters that define this maneuver. The cost function is given by

$$J = \min \sum_{i=1}^{n} \Delta v^i \tag{1}$$

where Δv^i is the sum of the two maneuvers for each spacecraft. The optimization parameters are

$$\boldsymbol{z}^T = \left(\tau_0 \quad \Delta \boldsymbol{v}_0 \quad \tau_m^1 \quad \cdots \quad \tau_m^n, T^1 \quad \cdots \quad T^n, \tau_f^1\right)$$

where τ_0 is the location on O^0 where the maneuver is applied, $\Delta \boldsymbol{v}_0$ is composed of the $\boldsymbol{ijk}$ components of the booster injection maneuver, τ_m^i is the location on O^m from which the individual spacecraft depart, T^i is the time of flight for each spacecraft between the departure maneuver from O^m and the arrival maneuver on O^f, and τ^1 is the arrival point of spacecraft 1 on O^f.

The injection maneuver is represented more conveniently in a frame that moves and rotates with the spacecraft. One convenient frame is the $\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}\hat{\boldsymbol{w}}$ frame defined by the basis vectors,

$$\hat{\boldsymbol{v}} \equiv \frac{\boldsymbol{v}}{|\boldsymbol{v}|}, \qquad \hat{\boldsymbol{w}} \equiv \frac{(\boldsymbol{r} \times \boldsymbol{v})}{|\boldsymbol{r} \times \boldsymbol{v}|}, \qquad \hat{\boldsymbol{u}} \equiv \hat{\boldsymbol{w}} \times \hat{\boldsymbol{v}}$$

This represents a vehicle centered osculating frame to which the angular direction of the maneuvers are referred. If Δv_0, α, and β are specified then

$$\Delta \boldsymbol{v}_0' = \begin{pmatrix} \Delta v_{0x}' \\ \Delta v_{0y}' \\ \Delta v_{0z}' \end{pmatrix} = \begin{pmatrix} \Delta v_0 \cos\alpha \cos\beta \\ \Delta v_0 \sin\alpha \cos\beta \\ \Delta v_0 \sin\beta \end{pmatrix}$$

where the prime symbol denotes quantities referred to in the $\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}\hat{\boldsymbol{w}}$ frame. Here, α is the right ascension of the $\Delta \boldsymbol{v}$ vector in the $\hat{\boldsymbol{v}}$–$\hat{\boldsymbol{u}}$ plane and β is the declination measured relative to the same plane. The $\boldsymbol{ijk}$ components of the maneuver are then determined by means of the transformation

$$\Delta \boldsymbol{v}_0 = R \Delta \boldsymbol{v}_0'$$

where R is the direction cosine matrix of the $\boldsymbol{vuw} \rightarrow \boldsymbol{ijk}$ transformation and is given by

$$R = \begin{bmatrix} \boldsymbol{i}\cdot\hat{\boldsymbol{v}}' & \boldsymbol{i}\cdot\hat{\boldsymbol{u}}' & \boldsymbol{i}\cdot\hat{\boldsymbol{w}}' \\ \boldsymbol{j}\cdot\hat{\boldsymbol{v}}' & \boldsymbol{j}\cdot\hat{\boldsymbol{u}}' & \boldsymbol{j}\cdot\hat{\boldsymbol{w}}' \\ \boldsymbol{k}\cdot\hat{\boldsymbol{v}}' & \boldsymbol{k}\cdot\hat{\boldsymbol{u}}' & \boldsymbol{k}\cdot\hat{\boldsymbol{w}}' \end{bmatrix}$$

The magnitude of the injection maneuver is bounded by the inequality constraint,

$$0 \leq \Delta v_0 \leq \Delta v_{0\max}$$

This constraint is removed when a slack variable a is introduced such that

$$\Delta v_0 = \Delta v_{0\max} \sin^2 a$$

where a is unbounded. In this way, the transformed vector of optimization parameters is given by

$$\boldsymbol{z}^T = \left(\tau_0 \quad a \quad \alpha \quad \beta, \tau_m^1 \quad \cdots \quad \tau_m^n, T^1 \quad \cdots \quad T^n, \tau_f^1\right)$$

resulting in a $2n + 5$ dimensional parameter optimization problem. Note that all of the boundary conditions and constraints are satisfied at each value of $\boldsymbol{z}$ so that from an optimization algorithm point of view, the problem is unconstrained. This is because all of the required intermediate solutions of the Lambert problem are solved at each iteration.

The development of the cost gradient follows the same procedure as that required for the no-booster case[1] except that a modification is required for accounting for the fact that the intermediate transfer orbit O^m is not fixed because it depends on the variables that define the initial injection maneuver and location. The minimization of the characteristic velocity for transferring spacecraft i from O^m to O^f by means of two impulses requires minimizing the cost function given by

$$J^i = \Delta v_m + \Delta v_f$$

The total cost for the n spacecraft is

$$J = \sum_{i=1}^{n} J^i$$

For spacecraft i, when the components of the initial and the final maneuvers are treated as independent arguments of J^i, the differential of J^i is

$$\begin{aligned} \mathrm{d}J^i &= \frac{\partial \Delta v_m}{\partial \Delta \boldsymbol{v}_m} \mathrm{d}(\Delta \boldsymbol{v}_m) + \frac{\partial \Delta v_f}{\partial \Delta \boldsymbol{v}_f} \mathrm{d}(\Delta \boldsymbol{v}_f) \\ &= \frac{\Delta \boldsymbol{v}_m^T}{\Delta v_m}\left(\mathrm{d}\boldsymbol{v}_m^+ - \mathrm{d}\boldsymbol{v}_m^-\right) + \frac{\Delta \boldsymbol{v}_f^T}{\Delta v_f}\left(\mathrm{d}\boldsymbol{v}_f^+ - \mathrm{d}\boldsymbol{v}_f^-\right) \end{aligned}$$

where the $-$ and the $+$ signs indicate quantities before and after an impulse time, respectively. It can be shown that the generalized cost gradient for spacecraft i is[3]

$$\mathrm{d}J^i = \left[\left(\dot{\boldsymbol{p}}_m^T + \boldsymbol{p}_m^T H\right)\delta \boldsymbol{r}_m^- - \boldsymbol{p}_m^T \delta \boldsymbol{v}_m^-\right] + H_m^- \,\mathrm{d}\tau_m + H \,\mathrm{d}T + H_f^+ \,\mathrm{d}\tau_f \tag{2}$$

where $\boldsymbol{p}$ is the primer vector, $\dot{\boldsymbol{p}}$ is the primer rate, and H is the Hamiltonian given by

$$H = \dot{\boldsymbol{p}}^T \boldsymbol{v} - \boldsymbol{p}^T \boldsymbol{g}$$

where $\boldsymbol{v}$ is the velocity and $\boldsymbol{g}$ is the gravitational force-field vector. Subscripts denote the times at which these values are evaluated, where m corresponds to t_m and f corresponds to t_f. The term in brackets is referred to as the adjoint equation for a system with velocity-dependent terms and is required because the intermediate orbit O^m changes as a result of perturbations in both $\Delta \boldsymbol{v}_0$ and $\mathrm{d}\tau_0$, which in turn introduce the nonzero perturbations $\delta \boldsymbol{r}_m^-$ and $\delta \boldsymbol{v}_m^-$ at t_m on O^m. If O^m remains fixed, then this term is zero and the cost gradient is the same as that for the standard orbit transfer problem. Here $\delta \boldsymbol{r}_m^-$ and $\delta \boldsymbol{v}_m^-$ are contemporaneous position and velocity perturbations, respectively, before the initial spacecraft maneuver on O^m. These are related to the state variations along O^m by means of the 6×6 state transition matrix $\Phi(t_m, t_0)$, evaluated along this orbit from t_0 to t_m:

$$\delta \boldsymbol{x}_m^- = \Phi(t_m, t_0)\delta \boldsymbol{x}_0^+, \qquad \begin{pmatrix} \delta \boldsymbol{r}_m^- \\ \delta \boldsymbol{v}_m^- \end{pmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}_{(t_m, t_0)} \begin{pmatrix} \delta \boldsymbol{r}_0^+ \\ \delta \boldsymbol{v}_0^+ \end{pmatrix}$$

Solving for $\delta \boldsymbol{r}_m^-$ and $\delta \boldsymbol{v}_m^-$ and inserting into the expression for $\mathrm{d}J^i$ yields,

$$\begin{aligned} \mathrm{d}J^i &= \dot{\boldsymbol{q}}_m^T\left(\Phi_{11}\delta \boldsymbol{r}_0^+ + \Phi_{12}\delta \boldsymbol{v}_0^+\right) - \boldsymbol{p}_m^-\left(\Phi_{21}\delta \boldsymbol{r}_0^+ + \Phi_{22}\delta \boldsymbol{v}_0^+\right) \\ &\quad + H_m^- \,\mathrm{d}\tau_m + H \,\mathrm{d}T + H_f^+ \,\mathrm{d}\tau_f \qquad (3) \\ &= \left(\dot{\boldsymbol{q}}_m^T \Phi_{11} - \boldsymbol{p}_m^- \Phi_{21}\right)\left(\delta \boldsymbol{r}_0^+\right) + \left(\dot{\boldsymbol{q}}_m^T \Phi_{12} - \boldsymbol{p}_m^T \Phi_{22}\right)\left(\delta \boldsymbol{v}_0^+\right) \\ &\quad + H_m^- \,\mathrm{d}\tau_m + H \,\mathrm{d}T + H_f^+ \,\mathrm{d}\tau_f \qquad (4) \end{aligned}$$

where

$$\dot{\boldsymbol{q}}_m^T \equiv \dot{\boldsymbol{p}}_m^T + \boldsymbol{p}_m^T H$$

At the injection point the position and the velocity time-free variations (noncontemporaneous perturbations) are given by

$$\mathrm{d}\boldsymbol{r}_0^+ = \delta\boldsymbol{r}_0^+ + \boldsymbol{v}_0^+\,\mathrm{d}t_0, \qquad \mathrm{d}\boldsymbol{r}_0^- = \delta\boldsymbol{r}_0^- + \boldsymbol{v}_0^-\,\mathrm{d}\tau_0$$

$$\mathrm{d}\boldsymbol{v}_0^+ = \delta\boldsymbol{v}_0^+ + \boldsymbol{f}_0^+\,\mathrm{d}t_0, \qquad \mathrm{d}\boldsymbol{v}_0^- = \delta\boldsymbol{v}_0^- + \boldsymbol{f}_0^-\,\mathrm{d}\tau_0$$

where $\boldsymbol{f}_0^-$ and $\boldsymbol{f}_0^+$ are the vector force fields before and after the injection point, respectively. Because the initial orbit O^0 is fixed,

$$\delta\boldsymbol{r}_0^- = 0, \qquad \delta\boldsymbol{v}_0^- = 0$$

Continuity of the position and the vector field across the injection impulse requires that

$$\mathrm{d}\boldsymbol{r}_0^- = \mathrm{d}\boldsymbol{r}_0^+ = \mathrm{d}\boldsymbol{r}_0, \qquad \boldsymbol{f}_0^- = \boldsymbol{f}_0^+ = \boldsymbol{f}_0$$

so that

$$\delta\boldsymbol{r}_0^+ = \boldsymbol{v}_0^-\,\mathrm{d}\tau_0 - \boldsymbol{v}_0^+\,\mathrm{d}t_0$$

$$\begin{aligned}\delta\boldsymbol{v}_0^+ &= \boldsymbol{f}_0^-\,\mathrm{d}\tau_0 - \boldsymbol{f}_0^+\,\mathrm{d}t_0 + \left(\mathrm{d}\boldsymbol{v}_0^+ - \mathrm{d}\boldsymbol{v}_0^-\right)\\ &= \boldsymbol{f}_0^-\,\mathrm{d}\tau_0 - \boldsymbol{f}_0^+\,\mathrm{d}t_0 + \mathrm{d}(\Delta\boldsymbol{v}_0)\end{aligned}$$

Fixing the initial time so that $\mathrm{d}t_0 = 0$ results in

$$\delta\boldsymbol{r}_0^+ = \boldsymbol{v}_0^-\mathrm{d}\tau_0, \qquad \delta\boldsymbol{v}_0^+ = \boldsymbol{f}_0^-\,\mathrm{d}\tau_0 + \mathrm{d}(\Delta\boldsymbol{v}_0)$$

and it remains to determine $\mathrm{d}(\Delta\boldsymbol{v}_0)$ as a function of the variations of the parameters used to describe $\Delta\boldsymbol{v}_0$. If the components used to specify the injection maneuver are grouped as

$$\boldsymbol{b}'^T = (a \quad \alpha \quad \beta)$$

then the differential of the injection maneuver is

$$\begin{aligned}\mathrm{d}(\Delta\boldsymbol{v}_0) &= \frac{\partial\Delta\boldsymbol{v}_0}{\partial\Delta\boldsymbol{v}_0'}\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}\,\mathrm{d}(\boldsymbol{b}') + \frac{\partial\Delta\boldsymbol{v}_0}{\partial\tau_0}\,\mathrm{d}\tau_0\\ &= R\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}\,\mathrm{d}(\boldsymbol{b}') + \left(\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-}\frac{\partial\boldsymbol{r}_0^-}{\partial\tau_0}\,\mathrm{d}\tau_0 + \frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-}\frac{\partial\boldsymbol{v}_0^-}{\partial\tau_0}\,\mathrm{d}\tau_0\right)\\ &= R\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}\,\mathrm{d}(\boldsymbol{b}') + \left(\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-}\boldsymbol{v}_0^- + \frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-}\boldsymbol{g}_0^-\right)\mathrm{d}\tau_0\end{aligned}$$

Note that the variation with respect to τ_0 is required because the $\boldsymbol{vuw}$ frame in which $\boldsymbol{b}'$ is defined is dependent on the state at t_0 and the location on O^0 given by τ_0, i.e., a perturbation in τ_0 will produce an independent perturbation in $\Delta\boldsymbol{v}_0$. The gradients of the injection maneuver with respect to the position and the velocity vectors before the injection maneuver are given by

$$\begin{aligned}\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-} &= \frac{\partial R}{\partial\boldsymbol{r}_0^-}\Delta\boldsymbol{v}_0'\\ &= \left(\frac{\partial\hat{\boldsymbol{v}}}{\partial\boldsymbol{r}_0^-}\Delta\boldsymbol{v}_0^- \,\middle|\, \frac{\partial\hat{\boldsymbol{u}}}{\partial\boldsymbol{r}_0^-}\Delta\boldsymbol{v}_0^- \,\middle|\, \frac{\partial\hat{\boldsymbol{w}}}{\partial\boldsymbol{r}_0^-}\Delta\boldsymbol{v}_0^-\right)_{3\times 3}\\ \frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-} &= \frac{\partial R}{\partial\boldsymbol{v}_0^-}\Delta\boldsymbol{v}_0'\\ &= \left(\frac{\partial\hat{\boldsymbol{v}}}{\partial\boldsymbol{v}_0^-}\Delta\boldsymbol{v}_0^- \,\middle|\, \frac{\partial\hat{\boldsymbol{u}}}{\partial\boldsymbol{v}_0^-}\Delta\boldsymbol{v}_0^- \,\middle|\, \frac{\partial\hat{\boldsymbol{w}}}{\partial\boldsymbol{v}_0^-}\Delta\boldsymbol{v}_0^-\right)_{3\times 3}\end{aligned}$$

Recalling the definition of the unit vectors of the $\boldsymbol{vuw}$ frame, $\hat{\boldsymbol{v}} \equiv \boldsymbol{v}/v$, $\hat{\boldsymbol{w}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{w}} \times \hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$, we find that the gradients of these unit vectors with respect to the state vector $(\boldsymbol{r}_0, \boldsymbol{v}_0)$ at t_0^- are

$$\frac{\partial\hat{\boldsymbol{v}}}{\partial\boldsymbol{r}_0^-} = 0, \qquad \frac{\partial\hat{\boldsymbol{v}}}{\partial\boldsymbol{v}_0^-} = \bar{V}, \qquad \frac{\partial\hat{\boldsymbol{u}}}{\partial\boldsymbol{r}_0^-} = (\bar{R} \times \hat{\boldsymbol{v}}) \times \hat{\boldsymbol{v}}$$

$$\frac{\partial\hat{\boldsymbol{u}}}{\partial\boldsymbol{v}_0^-} = (\hat{\boldsymbol{r}} \times \bar{V}) \times \hat{\boldsymbol{v}} + (\hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}}) \times \bar{V}$$

$$\frac{\partial\hat{\boldsymbol{w}}}{\partial\boldsymbol{r}_0^-} = \bar{R} \times \hat{\boldsymbol{v}}, \qquad \frac{\partial\hat{\boldsymbol{w}}}{\partial\boldsymbol{v}_0^-} = \hat{\boldsymbol{r}} \times \bar{V}$$

where

$$\bar{R} \equiv (1/|\boldsymbol{r}|)(I - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}^T), \qquad \bar{V} \equiv (1/|\boldsymbol{v}|)(I - \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T)$$

and I is a 3×3 identity matrix. Note also that both a matrix–vector cross product and a vector–matrix cross product have been used. For example, the matrix–vector product operation for a 3×3 matrix L and a 3×1 vector $\boldsymbol{k}$ is given by

$$L \times \boldsymbol{k} \equiv \left[(\boldsymbol{l}^1 \times \boldsymbol{k})_{3\times 1} \quad (\boldsymbol{l}^2 \times \boldsymbol{k})_{3\times 1} \quad (\boldsymbol{l}^3 \times \boldsymbol{k})_{3\times 1}\right]_{3\times 3}$$

where $\boldsymbol{l}^i$ is the ith column of L. The product $L \times \boldsymbol{k}$ is thus a 3×3 matrix whose columns are the vectors that result from the cross products of the columns of L and the vector $\boldsymbol{k}$. The vector–matrix product is defined in an analogous way, except that the order of the cross-product terms is reversed:

$$\boldsymbol{k} \times L \equiv \left[(\boldsymbol{k} \times \boldsymbol{l}^1)_{3\times 1} \quad (\boldsymbol{k} \times \boldsymbol{l}^2)_{3\times 1} \quad (\boldsymbol{k} \times \boldsymbol{l}^3)_{3\times 1}\right]_{3\times 3}$$

In terms of τ_0 and $\boldsymbol{b}'$, the velocity perturbation after the injection maneuver is given by

$$\delta\boldsymbol{v}_0^+ = \left[\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-}\boldsymbol{v}_0^- + \left(I + \frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-}\right)\boldsymbol{g}_0^-\right]\mathrm{d}\tau_0 + R\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}\,\mathrm{d}\boldsymbol{b}'$$

where

$$\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'} = \begin{bmatrix}\dfrac{\partial\Delta v_{0x}}{\partial a}\cos\alpha\cos\beta & -\Delta v_0\sin\alpha\cos\beta & -\Delta v_0\cos\alpha\sin\beta\\ \dfrac{\partial\Delta v_{0x}}{\partial a}\sin\alpha\cos\beta & \Delta v_0\cos\alpha\cos\beta & -\Delta v_0\sin\alpha\sin\beta\\ \dfrac{\partial\Delta v_{0x}}{\partial a}\sin\beta & 0 & \Delta v_0\cos\beta\end{bmatrix}$$

$$\frac{\partial\Delta v_{0x}}{\partial a} = 2\Delta v_{0\max}\sin a\cos a$$

In terms of the injection maneuver variations $\mathrm{d}\boldsymbol{b}'$, the departure point on O^m, the transfer time between O^m and O^f, and the arrival point of O^f, the cost gradient for each spacecraft i is now given by

$$\mathrm{d}J^i = +\frac{\partial J^i}{\partial\tau_0}\,\mathrm{d}\tau_0 + \frac{\partial J^i}{\partial\boldsymbol{b}'}\,\mathrm{d}\boldsymbol{b}' + H_m^-\,\mathrm{d}\tau_m + H\,\mathrm{d}T + H_f^+\,\mathrm{d}\tau_f \qquad (i = 1, \ldots, n)$$

If the following two quantities are defined,

$$\boldsymbol{m} \equiv \left(\dot{\boldsymbol{q}}_m^T\Phi_{11} - \boldsymbol{p}_m^T\Phi_{21}\right), \qquad \boldsymbol{n} \equiv \left(\dot{\boldsymbol{q}}_m^T\Phi_{12} - \boldsymbol{p}_m^T\Phi_{22}\right)$$

then

$$\frac{\partial J^i}{\partial\tau_0} = \left\{\boldsymbol{m}\boldsymbol{v}_0^- + \boldsymbol{n}\left[\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-}\boldsymbol{v}_0^- + \left(I + \frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-}\right)\boldsymbol{f}_0^-\right]\right\}$$

$$\frac{\partial J^i}{\partial\boldsymbol{b}'} = \boldsymbol{n}R\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}$$

Generalizing the result to n spacecraft requires setting

$$\sum_{i=1}^{n}\frac{\partial J^i}{\partial\tau_0}=\left(\sum_{i=1}^{n}\boldsymbol{m}^i\right)\boldsymbol{v}_0^- + \left(\sum_{i=1}^{n}\boldsymbol{n}^i\right)\left[\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-}\boldsymbol{v}_0^- + \left(I+\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-}\right)\boldsymbol{f}_0^-\right]$$

$$\sum_{i=1}^{n}\frac{\partial J^i}{\partial\boldsymbol{b}'}=\left(\sum_{i=1}^{n}\boldsymbol{n}^i\right)R\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}$$

Finally, when the differentials of the final arrival-point constraints are eliminated, the differential of the cost for the weak-booster case is given by

$$\begin{aligned}\mathrm{d}J &= +T_{p0}\left\{\left(\sum_{i=1}^{n}\boldsymbol{m}^i\right)\boldsymbol{v}_0^- + \left(\sum_{i=1}^{n}\boldsymbol{n}^i\right)\left[\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{r}_0^-}\boldsymbol{v}_0^- + \left(I+\frac{\partial\Delta\boldsymbol{v}_0}{\partial\boldsymbol{v}_0^-}\right)\boldsymbol{f}_0^-\right]\right\}\mathrm{d}\tau_0\\ &+ \left(\sum_{i=1}^{n}\boldsymbol{n}^i\right)R\frac{\partial\Delta\boldsymbol{v}_0'}{\partial\boldsymbol{b}'}\begin{pmatrix}\mathrm{d}a\\ \mathrm{d}\alpha\\ \mathrm{d}\beta\end{pmatrix} + T_{pm}\left(H_m^{-1}+\sum_{i=2}^{n}H_f^{+i}\right)\mathrm{d}\tau_m^1\\ &+ T_{pm}\sum_{i=2}^{n}\left(H_m^{-i}-H_f^{+i}\right)\mathrm{d}\tau_m^i + \left(H^1+\sum_{i=2}^{n}H_f^{+i}\right)\mathrm{d}T^1\\ &+ \sum_{i=2}^{n}\left(H^i-H_f^{+i}\right)\mathrm{d}T^i - T_{pf}\left(\sum_{i=1}^{n}H_f^{+i}\right)\mathrm{d}\tau_f^1 \qquad (5)\end{aligned}$$

The coefficients of the variations in the independent parameters are then used to form the gradient vector in an unconstrained parameter optimization algorithm. For each value of the parameter vector, the booster maneuver is made and the intermediate orbit O^m is computed. Then a series of generalized Lambert problems is solved for each spacecraft between the departure point on O^m and the arrival point on O^f.

The departure location on O^m for each spacecraft is given by τ_m^i. An additional constraint that may be needed and has not been used is that $\tau_m^i \geq 0$. This simply means that the departure points on O^m for each individual spacecraft must occur after the booster maneuver, because otherwise the departure points would be undefined. These inequality constraints can be removed when a real vector of slack variables $\boldsymbol{d}^i$ is introduced such that

$$(\boldsymbol{d}^i)^2 - \tau_m^i = 0 \qquad (i = 1,\ldots,n)$$

The implementation of these constraints is required only in the event that the algorithm is converging to negative values of τ_m^i.

Case III: The Strong-Booster Case

If a booster propellant constraint is not imposed, then it is possible for the transfer orbit to reach and thus intersect the final orbit at least once. In this case, it is possible for at least one spacecraft to complete the transfer by performing only one maneuver. If the transfer orbit intersects the final orbit more than once, it may even be possible for the other spacecraft to perform only one maneuver to complete the transfer as well. This occurrence would, however, depend on the synodic periods related to the initial orbit, the intermediate transfer orbit, and the final orbit. In general, however, it can be assumed that one spacecraft performs a single maneuver, whereas the other spacecraft perform the required number of maneuvers to complete a feasible transfer.

The objective is to minimize the total sum of all of the velocity impulse magnitudes made by all of the spacecraft; the velocity magnitude of the booster maneuver is not part of the cost function, but the parameters that define the boost maneuver are used as part of the optimization variables. With this, the cost function is

$$J=\sum_{i=1}^{n}\Delta v_T^i$$

where Δv_T^i is the sum of the magnitudes of the maneuvers made by spacecraft i:

$$\Delta v_T^i=\sum_{j=1}^{m}\Delta v_j$$

and m is the number of maneuvers assumed for the particular spacecraft. For simplicity, it will be assumed that spacecraft 2 to n will perform at most two maneuvers; spacecraft 1 makes only one maneuver, as the booster trajectory is on a path that intersects the final orbit. The set of optimization variables is then given by the parameter vector,

$$z^T=\begin{bmatrix}\tau_0^B & T^1 & \tau_f^1, \tau_m^2 & \tau_m^3 & \cdots & \tau_m^n, T^2 & T^3 & \cdots & T^n\end{bmatrix}$$

where τ_0^B represents the departure point for the booster on orbit O^0, T^1 is the transfer time between the booster maneuver at τ_0^B and spacecraft 1's maneuver at location τ_f^1 on O^f. The booster trajectory and spacecraft 1's trajectory are the same by definition, and this will be referred to as the intermediate orbit O^m. The quantities τ_m^i $(i=2,\ldots,n)$ represent the departure points for spacecraft $2,\ldots,n$ from O^m. The quantities T^i $(i=2,\ldots,n)$ are the transfer times for each spacecraft between the departure maneuver on O^m and the arrival maneuver on O^f. The quantities τ_f^i $(i=2,\ldots,n)$ represent the arrival locations on the final orbit for each spacecraft. These quantities are not independent, as they are constrained by the spacing constraint. The $n-1$ spacing constraints for this case are given by

$$\tau_f^j=\tau_f^1+\frac{T_{pm}}{T_{pf}}\tau_m^j+\frac{1}{T_{pf}}(T^j-T^1)+(j-1)\Delta\tau \qquad j=2,\ldots,n \qquad (6)$$

where T_{pm} and T_{pf} are characteristic time-length units on the intermediate and the final orbits, respectively, which are taken to be the time periods of the orbits if they are periodic. If the orbits are not periodic, then some suitable time span encompassing the orbit arc can be used. The parameter $\Delta\tau$ is the required spacing between spacecraft. Note that the booster injection maneuver on O^0 is determined uniquely by means of a solution to the generalized Lambert problem defined by τ_0^B, T^1, and τ_f^1, and these replace the standard injection parameters Δv_0, α, and β in the parameter vector.

The cost gradient for spacecraft 1 is equivalent to the differential cost function of transferring a single spacecraft from O^0 to O^f, for which the departure maneuver cost is unimportant and therefore not included in the cost function. Note that it is the booster that makes this maneuver so that this transfer is the same for the booster and spacecraft 1. For spacecraft 1, $\mathrm{d}J$ is given by

$$\mathrm{d}J^1=H_0^-T_{p0}\,\mathrm{d}\tau_0+H\,\mathrm{d}T-H_f^+T_{pf}\,\mathrm{d}\tau_f$$

where H is the Hamiltonian:

$$H=\dot{\boldsymbol{p}}^T\boldsymbol{v}-\boldsymbol{p}^T\boldsymbol{g}$$

where $\dot{\boldsymbol{p}}$ is the primer rate vector, $\boldsymbol{p}$ is the primer vector, $\boldsymbol{v}$ is the velocity, and $\boldsymbol{g}$ is the gravitational force-field vector. The subscripts on H denote where it is evaluated: H_0^- is evaluated on the initial orbit before the departure maneuver, H is evaluated along the transfer after the departure maneuver, and H_f^- is evaluated on the final orbit after the final maneuver. The boundary conditions on the primer vector are given by the following necessary conditions[3]:

$$\boldsymbol{p}_0=0, \qquad \boldsymbol{p}_f=\Delta\boldsymbol{v}_f/\Delta v_f$$

For the remaining $n-1$ spacecraft, the cost gradient for each is given by

$$\mathrm{d}J^i = \left[\left(\dot{\boldsymbol{p}}_m^T + \boldsymbol{p}_m^T H\right)\delta \boldsymbol{r}_m^- - \boldsymbol{p}_m^T \delta \boldsymbol{v}_m^-\right] + H_m^- T_{pm}\, \mathrm{d}\tau_m + H\, \mathrm{d}T + H_f^+ T_{pf}\, \mathrm{d}\tau_f \tag{7}$$

where the subscript m refers to the time index at τ_m, which is the departure point on O^m. Because the remaining spacecraft are required to transfer from O^m to O^f, the boundary conditions on the primer vector associated with each of these are

$$\boldsymbol{p}_m^i = \Delta \boldsymbol{v}_m / \Delta v_m, \qquad \boldsymbol{p}_f^i = \Delta \boldsymbol{v}_f / \Delta v_f \qquad (i = 2, \ldots, n)$$

The total differential of the cost function is then given by

$$\mathrm{d}J = \mathrm{d}J^1 + \sum_{i=2}^{n} \mathrm{d}J^i$$

where it is noted that part of the optimization variables [the final arrival points τ_f^i $(i = 2, \ldots, n)$] are not independent. The $\mathrm{d}\tau_f^i$ $(i = 2, \ldots, n)$ are removed by means of the differentials of the final point spacing constraints:

$$\mathrm{d}\tau_f^i = \mathrm{d}\tau_f^1 + \frac{T_{pm}}{T_{pf}}\, \mathrm{d}\tau_m^i + \frac{1}{T_{pf}}(\mathrm{d}T^i - \mathrm{d}T^1) \qquad i = 2, \ldots, n$$

The expression for the adjoint equation at the point of departure on O^m associated with spacecrafts $2, \ldots, n$ is

$$\left[\left(\dot{\boldsymbol{p}}_m^T + \boldsymbol{p}_m^T H\right)\delta \boldsymbol{r}_m^- - \boldsymbol{p}_m^T \delta \boldsymbol{v}_m^-\right]^i$$

These are dependent on the three parameters used to define the booster intermediate orbit, which is the same orbit for spacecraft 1 and will be referred to by τ_0, T, τ_f, where

$$\tau_0 = \tau_0^B, \qquad T = T^1, \qquad \tau_f = \tau_f^1$$

It can be shown that the adjoint equation for spacecrafts $2, \ldots, n$ can be related to a perturbation in τ_0, T, τ_f as follows. On O^0, the time-free position and velocity perturbations at the departure point are

$$\mathrm{d}\boldsymbol{r}_0^- = \delta \boldsymbol{r}_0^- + \boldsymbol{v}_0^-\, \mathrm{d}\tau_0, \qquad \mathrm{d}\boldsymbol{r}_0^+ = \delta \boldsymbol{r}_0^+ + \boldsymbol{v}_0^+\, \mathrm{d}t_0$$

$$\mathrm{d}\boldsymbol{v}_0^- = \delta \boldsymbol{v}_0^- + \boldsymbol{f}_0^-\, \mathrm{d}\tau_0, \qquad \mathrm{d}\boldsymbol{v}_0^+ = \delta \boldsymbol{v}_0^+ + \boldsymbol{f}_0^+\, \mathrm{d}t_0$$

where $\boldsymbol{f}$ is the vector field. For the RTBP force-field model the vector field can be split between two components that depend exclusively on $\boldsymbol{r}$ and $\boldsymbol{v}$, i.e., $\boldsymbol{f} = \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v})$. Since both the initial orbit and the initial time index are fixed,

$$\delta \boldsymbol{r}_0^- = \delta \boldsymbol{v}_0^- = 0, \qquad \mathrm{d}t_0 = 0$$

then

$$\mathrm{d}\boldsymbol{r}_0^- = \boldsymbol{v}_0^-\, \mathrm{d}\tau_0, \qquad \mathrm{d}\boldsymbol{r}_0^+ = \delta \boldsymbol{r}_0^+, \qquad \mathrm{d}\boldsymbol{v}_0^- = \boldsymbol{f}_0^-\, \mathrm{d}\tau_0, \qquad \mathrm{d}\boldsymbol{v}_0^+ = \delta \boldsymbol{v}_0^+$$

because the time-free position perturbations are equivalent, i.e., $\mathrm{d}\boldsymbol{r}_0^- = \mathrm{d}\boldsymbol{r}_0^+ = \mathrm{d}\boldsymbol{r}_0$; then

$$\delta \boldsymbol{r}_0^+ = \boldsymbol{v}_0^-\, \mathrm{d}\tau_0 \tag{8}$$

At the arrival point, these perturbations take the form

$$\mathrm{d}\boldsymbol{r}_f^- = \delta \boldsymbol{r}_f^- + \boldsymbol{v}_f^-\, \mathrm{d}t_f, \qquad \mathrm{d}\boldsymbol{r}_f^+ = \delta \boldsymbol{r}_f^+ + \boldsymbol{v}_f^+\, \mathrm{d}\tau_f$$

$$\mathrm{d}\boldsymbol{v}_f^- = \delta \boldsymbol{v}_f^- + \boldsymbol{f}_f^-\, \mathrm{d}t_f, \qquad \mathrm{d}\boldsymbol{v}_f^+ = \delta \boldsymbol{v}_f^+ + \boldsymbol{f}_f^+\, \mathrm{d}\tau_f$$

The final orbit is fixed, but the final time or the transfer time is not, so that

$$\mathrm{d}\boldsymbol{r}_f^- = \delta \boldsymbol{r}_f^- + \boldsymbol{v}_f^-\, \mathrm{d}t_f, \qquad \mathrm{d}\boldsymbol{r}_f^+ = \boldsymbol{v}_f^+\, \mathrm{d}\tau_f$$

$$\mathrm{d}\boldsymbol{v}_f^- = \delta \boldsymbol{v}_f^- + \boldsymbol{f}_f^-\, \mathrm{d}t_f, \qquad \mathrm{d}\boldsymbol{v}_f^+ = \boldsymbol{f}_f^+\, \mathrm{d}\tau_f$$

Equating the final position perturbations and setting $t_f = T$ yield

$$\delta \boldsymbol{r}_f^- = \boldsymbol{v}_f^+\, \mathrm{d}\tau_f - \boldsymbol{v}_f^-\, \mathrm{d}T$$

The final position and velocity perturbations are also related to the initial position and velocity perturbations by means of the state-transition matrix evaluated from t_0 to t_f:

$$\begin{pmatrix} \delta \boldsymbol{r}_f^- \\ \delta \boldsymbol{v}_f^- \end{pmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}_{(t_f, t_0)} \begin{pmatrix} \delta \boldsymbol{r}_0^+ \\ \delta \boldsymbol{v}_0^+ \end{pmatrix}$$

from which

$$\delta \boldsymbol{v}_0^+ = \left(\Phi_{12}^{f0}\right)^{-1} \left(\delta \boldsymbol{r}_f^- - \Phi_{11}^{f0} \delta \boldsymbol{r}_0^+\right)$$

where

$$\Phi_{mn}^{ab} \equiv \Phi_{mn}(t_a, t_b)$$

Substituting the expressions for $\delta \boldsymbol{r}_0^+$ and $\delta \boldsymbol{r}_f^-$ into the expression for $\delta \boldsymbol{v}_0^+$ results in

$$\delta \boldsymbol{v}_0^+ = -\left[\left(\Phi_{12}^{f0}\right)^{-1} \Phi_{11}^{f0} \boldsymbol{v}_0^-\right] \mathrm{d}\tau_0 - \left[\left(\Phi_{12}^{f0}\right)^{-1} \boldsymbol{v}_f^-\right] \mathrm{d}T + \left[\left(\Phi_{12}^{f0}\right)^{-1} \boldsymbol{v}_f^+\right] \mathrm{d}\tau_f \tag{9}$$

so that finally both $\delta \boldsymbol{r}_0^+$ and $\delta \boldsymbol{v}_0^+$ have been expressed as functions of $\mathrm{d}\tau_0$, $\mathrm{d}T$, and $\mathrm{d}\tau_f$. The position and the velocity perturbations at t_m for spacecrafts $2, \ldots, n$ are also related to the perturbations at t_0:

$$\begin{pmatrix} \delta \boldsymbol{r}_m^- \\ \delta \boldsymbol{v}_m^- \end{pmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}_{(t_m, t_0)} \begin{pmatrix} \delta \boldsymbol{r}_0^+ \\ \delta \boldsymbol{v}_0^+ \end{pmatrix}$$

When this expression is used along with the expressions for $\delta \boldsymbol{r}_0^+$ and $\delta \boldsymbol{v}_0^+$, it can be shown that

$$\left(\dot{\boldsymbol{q}}_m^T \delta \boldsymbol{r}_m^- - \boldsymbol{p}_m^T \delta \boldsymbol{v}_m^-\right)^i = +\left[\boldsymbol{m} - \boldsymbol{n}\left(\Phi_{12}^{f0}\right)^{-1} \Phi_{11}^{f0}\right]^i \boldsymbol{v}_0^-\, \mathrm{d}\tau_0 - \left[\boldsymbol{n}\left(\Phi_{12}^{f0}\right)^{-1}\right]^i \boldsymbol{v}_f^-\, \mathrm{d}T + \left[\boldsymbol{n}\left(\Phi_{12}^{f0}\right)^{-1}\right]^i \boldsymbol{v}_f^+\, \mathrm{d}\tau_f \qquad (i = 2, \ldots, n)$$

where

$$\boldsymbol{m}^i \equiv \left(\dot{\boldsymbol{q}}_m^T \Phi_{11} - \boldsymbol{p}_m^T \Phi_{21}\right)^i, \qquad \boldsymbol{n}^i \equiv \left(\dot{\boldsymbol{q}}_m^T \Phi_{12} - \boldsymbol{p}_m^T \Phi_{22}\right)^i$$

When these last results are used and the dependent variations $\mathrm{d}\tau_f^i$ $(i = 2, \ldots, n)$ are eliminated, the differential of the cost for the strong-booster case is given by

$$\begin{aligned} \mathrm{d}J = {} & +T_{p0}\left\{H_0^{-1} + \sum_{i=2}^{n}\left[\boldsymbol{m} - \boldsymbol{n}\left(\Phi_{12}^{f0}\right)^{-1} \Phi_{11}^{f0}\right]^1\right\} \mathrm{d}\tau_0^B \\ & + \left\{H^1 - \sum_{i=2}^{n} H_f^{+i} - \sum_{j=2}^{n}\left[\boldsymbol{n}\left(\Phi_{12}^{f0}\right)^{-1}\right]^j \boldsymbol{v}_f^-\right\} \mathrm{d}T^1 \\ & - T_{pf}\left\{\sum_{i=1}^{n} H_f^{+i} + \sum_{j=2}^{n}\left[\boldsymbol{n}\left(\Phi_{12}^{f0}\right)^{-1}\right]^j \boldsymbol{v}_f^+\right\} \mathrm{d}\tau_f^1 \\ & + T_{pm}\sum_{i=2}^{n}\left(H_m^{-i} - H_f^{+i}\right) \mathrm{d}\tau_m^i + \sum_{i=2}^{n}\left(H^i - H_f^{+i}\right) \mathrm{d}T^i \end{aligned} \tag{10}$$

where H^i $(i = 2, \ldots, n)$ is the Hamiltonian evaluated between the impulses for spacecrafts 2 to n and the parameters defining the intermediate transfer orbit are reset to τ_0^B, T^1, and τ_f^1. The remaining variables have been defined above. The coefficients of the independent variations are used to form the cost gradient vector in an optimization algorithm. On an optimal transfer, $H_m^{-i} = H^i = H_f^{+i}$ for spacecrafts $2, \ldots, n$. The Hamiltonian for spacecraft 1 is related in a more complicated way to the other spacecrafts' value of the

Hamiltonian as a result of the extra terms in the coefficients of the variations of the booster transfer trajectory. The optimization of this type of transfer with the given cost gradient has been found to be very efficient and well behaved numerically.

The departure location on O^m for spacecraft 2 to n are given by τ_m^i. An additional constraint that may be needed and has not been used is that $\tau_m^i \geq 0$. As with the weak-booster case, these inequality constraints can be removed if necessary by introducing slack variables.

Transfer Examples

This section illustrates the use of the analytical cost gradient to optimize a nominal multiple-spacecraft transfer for three spacecraft with the weak-booster model. A parameter optimization algorithm is used to iterate on the unknown parameters to minimize the cost function. A minimum is found when the terms of the cost gradient vector are near zero and the Hessian matrix of the objective function is positive definite. The algorithm used in this study is the variable metric algorithm for unconstrained and constrained parameter optimization found in the routine VF13AD of the Harwell Subroutine Library.[9]

The circular RTBP is used to approximate the motion of spacecraft about the sun–Earth–moon system. The numerical parameters for this RTBP system are given in Table 1. The initial orbit and the final orbits are simple periodic distant retrograde orbits (DROs) about the Earth–moon system. In these examples, motion is confined to the (x, y) plane, which for the sun–Earth–moon RTBP model is the ecliptic plane. This assumption primarily simplifies the presentation of the results even though the formulation discussed in this analysis allows for motion in three dimensions. The parameters for the initial and the final orbits are given in Table 2.

Table 1 Sun–Earth–moon system RTBP parameters (International Astronautical Union, 1976)

Parameter	Symbol	Value	Units
Sun gravitational parameter	Gm_s	$1.32712438 \times 10^{20}$	m^3/s^2
Earth + moon gravitational parameter	Gm_e	$4.03503294 \times 10^{14}$	m^3/s^2
Distance unit (1 AU)	a	$1.49597870 \times 10^{11}$	m
Mean motion (calculated)	n	$1.99098670 \times 10^{-7}$	rad/s
RTBP mass ratio (calculated)	μ	$3.04042389 \times 10^{-6}$	——

Table 2 Initial and final orbit parameters for multiple-spacecraft transfer examples

Orbit	Description	x_0, km	$\dot{y}_0$, m/s	T_p, days
Initial, O^0	DRO	500,000	814.955	47.2921
Final, O^f	DRO	5,000,000	−2014.914	345.9780

Table 3 Booster and spacecraft parameters for weak-booster example

Parameter	Value
Booster initial mass	1000 kg
Booster burnout mass	850 kg
Booster specific impulse	300 s
Spacecraft initial mass (each)	1000 kg
Spacecraft specific impulse (each)	500 s

Transfer Example for the Weak-Booster Case

The optimal transfer solution for three spacecraft between DROs without the aid of a booster has been solved.[1] With the aid of a booster with a propellant constraint, it is anticipated that the final spacecraft masses on the final orbit will be greater than without the aid of the booster. The booster and the spacecraft parameters for this example are listed in Table 3. The booster parameters for this case were chosen to produce a transfer that would illustrate clearly the geometry of this type of transfer for the given orbits, and hence they do not necessarily correspond to a particular set of booster parameters for existing upper stages. The spacing requirement on the final orbit is 10% of the final orbit's time period, namely $\Delta\tau = 0.1$.

The results and the parameters for the nominal and the optimal transfer are given in Table 4. In this and other tables, the departure and the arrival points are given as fractions of the initial, intermediate, and final orbit periods, which have been normalized to unity, and

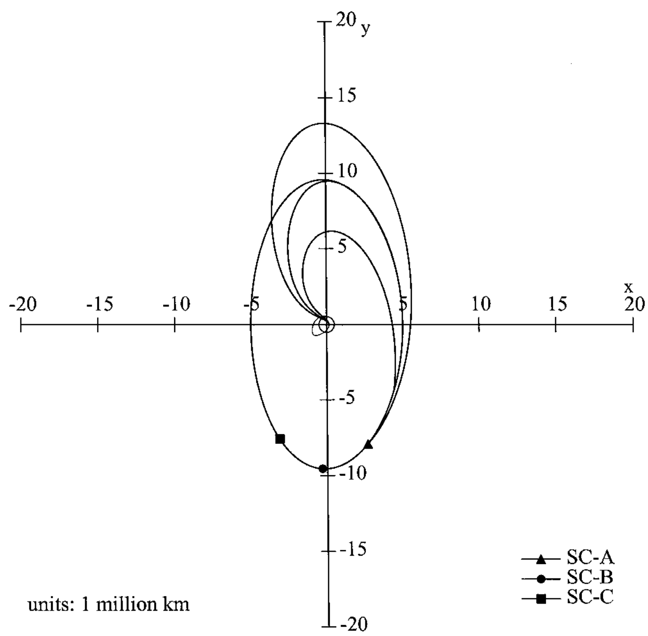

Fig. 1 Optimal transfer for the weak-booster case.

Table 4 Results for the weak-booster transfer case

Spacecraft separation, $\Delta\tau = 0.1$	Symbol	Nominal	Optimal	Units
Booster departure point	τ_0	1.5000	1.1297	Fraction O^0 period
Departure Δv slack variable	a	1.5700	1.5792	——
Departure Δv ascension angle	α	0.00	4.90	deg
Departure Δv declination angle	β	0.00	−0.34	deg
SC-A departure point	τ_m^A	1.0000	1.6733	Fraction O^m period
SC-B departure point	τ_m^B	1.0000	1.6625	Fraction O^m period
SC-C departure point	τ_m^C	1.0000	1.6469	Fraction O^m period
SC-A flight time	T^A	50.00	311.71	Days
SC-B flight time	T^B	50.00	205.54	Days
SC-C flight time	T^C	50.00	216.01	Days
SC-A arrival point	τ_f^A	0.6000	1.1555	Fraction O^f period
SC-A maneuver magnitude	Δv^A	2680.18	486.95	m/s
SC-B maneuver magnitude	Δv^B	3525.15	425.17	m/s
SC-C maneuver magnitude	Δv^C	4219.04	620.65	m/s
Total maneuver magnitude	Δv^T	10425.19	1532.77	m/s
SC-A final mass	m_f^A	578.91	905.46	kg
SC-B final mass	m_f^B	487.27	916.94	kg
SC-C final mass	m_f^C	422.97	881.11	kg

Table 5 Results for the strong-booster transfer case

Spacecraft separation, $\Delta\tau = 0.1$	Symbol	Nominal	Optimal	Units
Booster departure point	τ_0	1.1400	0.5621	Fraction O^0 period
Booster SC-A transfer time	T^A	300.00	205.91	Days
Booster SC-A arrival point	τ_f^A	1.0000	0.9412	Fraction O^f period
SC-B departure point	τ_m^B	0.9000	0.5042	Fraction O^m period
SC-C departure point	τ_m^C	0.9000	1.0159	Fraction O^m period
SC-B flight time	T^B	30.00	310.78	Days
SC-C flight time	T^C	30.00	180.06	Days
SC-A maneuver magnitude	Δv^A	680.66	136.42	m/s
SC-B maneuver magnitude	Δv^B	4890.00	318.61	m/s
SC-C maneuver magnitude	Δv^C	4270.66	267.53	m/s
Total maneuver magnitude	Δv^T	9841.14	722.58	m/s
SC-A final mass	m_f^A	870.39	972.56	kg
SC-B final mass	m_f^B	368.88	937.09	kg
SC-C final mass	m_f^C	418.54	946.90	kg

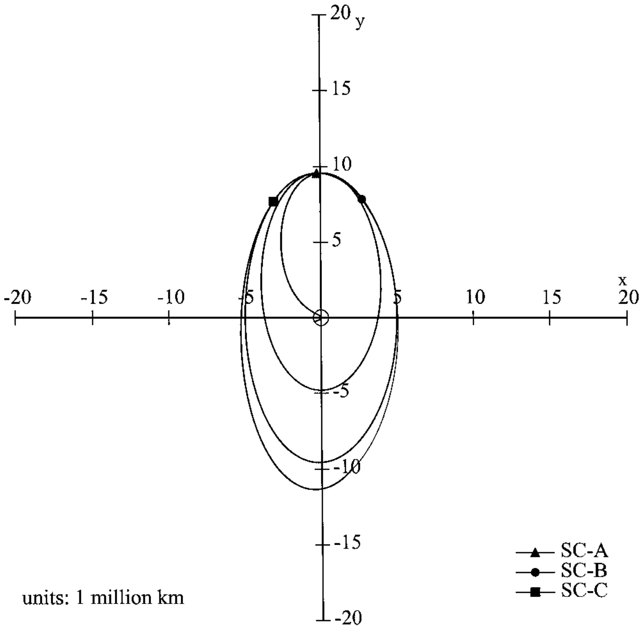

Fig. 2 Optimal transfer for the strong-booster case.

zero is the epoch at which the initial conditions are specified. The optimal transfer solution is shown in Fig. 1. An interesting result is that the booster places the spacecraft on a trajectory that has a close Earth flyby. The departure maneuvers for each of the spacecraft occur near perigee at different locations and hence these are powered Earth flybys. For this particular example, the cost decreased sharply from the nominal toward the optimal solution. However, near the minimum further iterations failed to decrease the cost and the iteration process was terminated. The cost became insensitive to some of the parameters and in fact, the departure Δv declination angle β was nonzero but small at the time the iterations were terminated. Had this transfer been constrained or modeled strictly as a planar transfer, β would have been zero.

Transfer Example for the Strong-Booster Case

The algorithm for the strong-booster case is used to compute the transfer for three spacecraft between the terminal orbits. For this case, the booster parameters are not required until after a solution is obtained in order to determine the class of booster that is required for the transfer.

The results and the parameters for the nominal and the optimal transfer are given in Table 5. The conditions that generate the nominal transfer are chosen such that the first spacecraft arrives on the final orbit at the intersection with the positive x axis; the remaining two spacecraft are required to depart the injection orbit some time before that. The optimal solution is shown in Fig. 2. Note that spacecraft A performs only one maneuver, namely the orbit insertion maneuver in which the booster transfer trajectory intersects the final orbit. As anticipated, this case yields the highest values for the final masses of the individual spacecraft.

Table 6 Orbit parameters for the MSIC transfer

Orbit	Description	x_0, km	$\dot{y}_0$, m/s	T_p, days
Initial, O^0	LEO	6,659.053	7,782.823	0.0622
Final, O^f	DRO	5,000,000	−2,014.914	345.9780

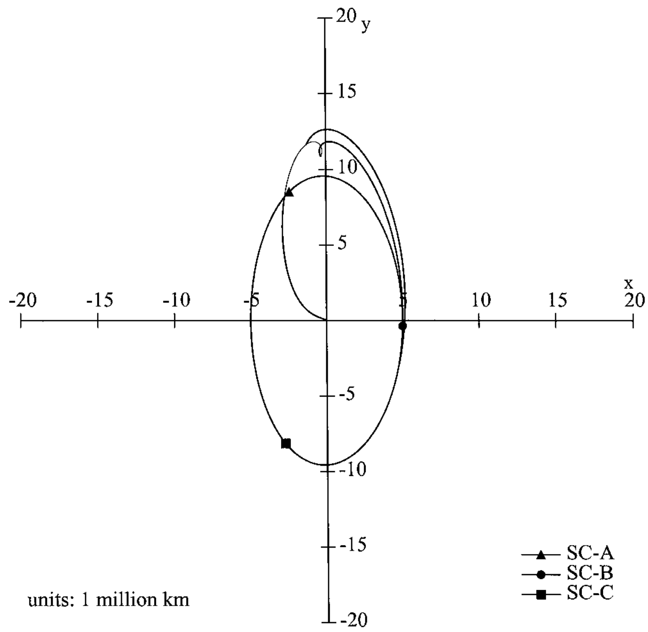

Fig. 3 Optimal transfer for a MSIC.

Transfers for a Multiple-Spacecraft Interferometer Constellation

This section presents the low Earth orbit (LEO)-DRO transfer of a three-spacecraft constellation required to be spaced out equally in time along a DRO about the Earth–moon system. There are a couple of space physics missions that are considering the use of multiple spacecraft to serve as an interferometer for the detection of gravity waves.[10,11] The example transfer presented here is similar to that required by the mission discussed by Hellings et al.[10] except that the final orbit in this case is the same DRO discussed in all of the preceding examples. This example is referred to as a multiple-spacecraft interferometer constellation (MSIC) transfer. The parameters describing the two terminal orbits are listed in Table 6. The initial spacecraft masses are 1000 kg, and the specific impulse for each is 500 s. To approximate and simulate the injection conditions from LEO, an approximate launch model used by

Table 7 MSIC strong-booster transfer results

Spacecraft separation, $\Delta\tau = \frac{1}{3}$	Symbol	Nominal	Optimal	Units
Booster departure point	τ_0	0.9750	0.9756	Fraction O^0 period
Booster SC-A transfer time	T^A	300.00	110.19	Days
Booster SC-A arrival point	τ_f^A	1.0000	0.6608	Fraction O^f period
SC-B departure point	τ_m^B	1.1000	0.8938	Fraction O^m period
SC-C departure point	τ_m^C	1.1000	0.6005	Fraction O^m period
SC-B flight time	T^B	100.00	192.16	Days
SC-C flight time	T^C	100.00	181.40	Days
SC-A maneuver magnitude	Δv^A	750.77	740.58	m/s
SC-B maneuver magnitude	Δv^B	1808.11	700.74	m/s
SC-C maneuver magnitude	Δv^C	2787.31	564.17	m/s
Total maneuver magnitude	Δv^T	5384.12	2005.48	m/s
SC-A final mass	m_f^A	858.03	859.82	kg
SC-B final mass	m_f^B	691.60	866.83	kg
SC-C final mass	m_f^C	566.40	891.31	kg

Pu and Edelbaum[12] is used in this study. A 200-km circular parking orbit about the Earth is approximated by assuming a circular parking orbit about the Earth + moon mass at an altitude at which the inertial circular orbit velocity is equal to the circular orbit velocity of the original 200-km circular orbit about the Earth alone. The spacing requirement is to space the three spacecraft evenly along the periodic orbit so that $\Delta\tau = \frac{1}{3}$.

If an upper stage is used to provide injection maneuver from LEO and the $\Delta v_{\max}$ capability of the booster is unconstrained, then the strong-booster case is the case that maximizes the spacecrafts' final mass on orbit. The results and parameters for the nominal and the optimal transfers are given in Table 7. The nominal transfer is selected by having the first spacecraft arrive on the DRO at the intersection with the positive x axis and the remaining spacecraft depart the injection orbit some time after that. The optimal solution is shown in Fig. 3. As expected, there is a large improvement on the final mass placed on orbit for each of the spacecraft. It is interesting to note that even though the first spacecraft (SC-A) performs only one maneuver, the magnitude of this maneuver is larger than the sum of the two maneuvers made by either SC-B or SC-C.

Conclusions

The weak- and the strong-booster cases of the multiple-spacecraft transfer problem have been presented. For a cost function that is taken to be the sum of all of the impulsive maneuvers made by all of the spacecraft and including the booster injection parameters as independent parameters, the differential of the cost was derived with primer vector theory. This provides analytical cost gradient equations that can be used in an unconstrained parameter optimization algorithm to obtain transfer solutions for n spacecraft equally spaced in time along a final orbit. In the current study, it may be possible to improve the overall performance of the transfer if more than two maneuvers are allowed per spacecraft. The formulation allows the addition of a further set of parameters to determine if intermediate impulses can lower the transfer cost for some or all of the spacecraft. Such a set would consist of positions and times for intermediate impulses, and the variation of the cost with respect to perturbations in these parameters can be derived from the same procedure presented here.

For the class of orbits used in the first example, the weak- and the strong-booster cases resulted in a transfer involving a close Earth flyby for all three spacecraft. The Δv costs for each of the three spacecraft were significantly different for both cases, and, as expected, the overall costs were higher for the weak-booster case. These equations and their implementation in a parameter optimization algorithm can be used to compute transfer strategies efficiently for multiple spacecraft launched on the same booster for which all spacecraft are targeted to be spaced along the same final orbit.

Acknowledgments

This work is part of C. A. Ocampo's graduate research at the University of Colorado, Boulder, during the period 1992–1996, which was supported by a NASA Graduate Student Researchers Program Fellowship. The support of the fellowship and the individuals associated with it is gratefully acknowledged.

References

[1] Ocampo, C. A., and Rosborough, G. W., "The Multiple Spacecraft Orbit Transfer Problem: The No Booster Case," *Journal of Guidance, Control, and Dynamics* (submitted for publication).

[2] Szebehely, V., *Theory of Orbits: The Restricted Problem of Three Bodies*, Academic, New York, 1967.

[3] Ocampo, C. A., "Trajectory Optimization for Distant Earth Satellites and Satellite Constellations," Ph.D. Dissertation, Dept. of Aerospace Engineering Sciences, Univ. of Colorado, Boulder, CO, Aug. 1996; University Microfilms Inc. (UMI), Publ. 9709513, Ann Arbor, MI.

[4] Lawden, D. E., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963.

[5] Marec, J. P., *Optimal Space Trajectories*, Elsevier Scientific, New York, 1979.

[6] Prussing, J. E., and Chiu, J. H., "Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 9, No. 1, 1986, pp. 17–22.

[7] Lion, P. M., and Handelsman, M., "Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, Vol. 6, No. 1, 1968, pp. 127–132.

[8] Jezewski, D. J., "Primer Vector Theory and Applications," NASA TR 454, NASA Johnson Space Center, Houston, TX, 1975.

[9] *The Harwell Subroutine Library Release 11*, Harwell Lab., Oxfordshire, England, UK, 1993.

[10] Hellings, R., Ciufolini, G., Giampieri, G., Lo, M., Gonzales, J., Marchal, C., Ocampo, C., Touboul, P., Robertson, D., Vaillon, L., "SAGITTARIUS: A Spaceborne Astronomical Gravity-Wave Interferometer," JPL Engineering Memorandum 314-569 (1993), Jet Propulsion Lab., Pasadena, California.

[11] Folkner, W. M., Bender, P. L., Stebbins, R. T., "LISA Mission Concept Study: Laser Interferometer Space Antenna For the Detection and Observation of Gravitational Waves," Jet Propulsion Lab. Publication, 97-16, Pasadena, CA, March 1998.

[12] Pu, C. L., and Edelbaum, T. N., "Four Body Trajectory Optimization," *AIAA Journal*, Vol. 13, No. 3, 1975, pp. 333–336.